UNIT - III

DISCRETIZATION

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Prandtl's Boundary Layer Equations

The Reynolds number has the form of a non-dimensional parameter

$$Re = \frac{LV}{\nu} = \frac{\rho L V}{\mu} \longrightarrow (1)$$

Where,

$L \rightarrow$ characteristic length, m.

$V \rightarrow$ Velocity of the flow, m/sec

$\nu \rightarrow$ Kinematic viscosity, m^2/sec

$\mu \rightarrow$ Dynamic viscosity, $N\text{-}S/m^2$

$\rho \rightarrow$ Density of the fluid, kg/m^3

The Reynolds number is the ratio of inertia to viscous forces following the "principle of similarity."

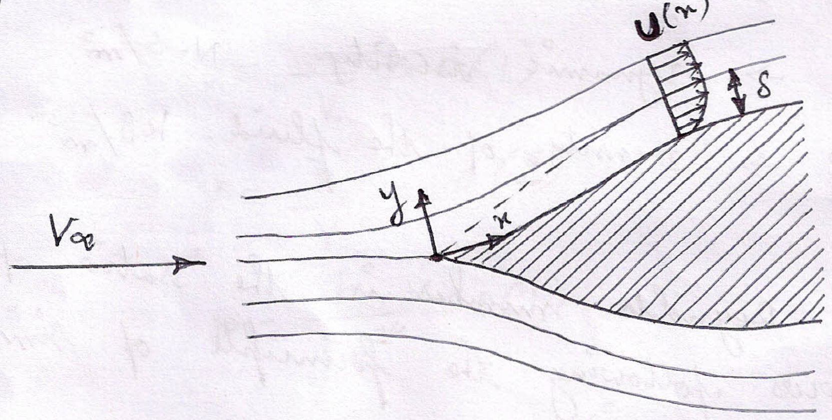
$$Re = \frac{\text{inertia force}}{\text{Viscous force}} = \frac{\rho u \left(\frac{\partial u}{\partial x}\right)}{\mu \left(\frac{\partial^2 u}{\partial x^2}\right)} \longrightarrow (2)$$

The velocity u at some point in the velocity field is proportional to the free stream velocity V. The velocity gradient $\frac{\partial u}{\partial x}$ is proportional to V/L and similarly $\frac{\partial^2 u}{\partial x^2}$ is proportional to $\frac{V}{L^2}$

Hence eqn ② becomes,

$$R_e = \frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho L V}{\mu}$$

Two flows are similar from the point of view of the relative importance of inertial and viscous ~~effects~~ effects if the Reynolds number is constant. The physical phenomenon of a flow with high Reynolds number is considered for the ~~explan~~ example of a cylindrical body.



Boundary layer flow along a wall.

With the exception of the immediate neighbourhood ~~for~~ of the surface, the flow velocity is comparable to the free stream velocity V. This flow region is nearly free of friction, it is a potential flow. considering the region near the surface there is friction in the flow which means that the fluid is retarded until it adheres at surface.

The transition from zero velocity at the surface to the full magnitude at some distance from it takes place in a very thin layer, the so-called 'boundary layer'. Its thickness is δ, which is a function of the downstream coordinate x and is assumed to be very small compared to the length of the body L.

In the normal direction y inside the thin layer it is clear that the gradient $\frac{\partial u}{\partial y}$ is very large compared to gradients in the streamwise direction $\partial u/\partial x$. The viscosity was meant to be very small in this flow, though the shear stress

$$\tau = \mu \frac{\partial u}{\partial y}$$ may assume large values

The above assumptions are used to simplify the Navier-Stokes equations for steady two-dimensional, laminar and incompressible flows. Including the continuity eqn. they have the following dimensional form in cartesian coordinates.

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\bar{\mu}}{\bar{\rho}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\bar{\mu}}{\bar{\rho}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right)$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

Here the velocity components $\bar{u}$ and $\bar{v}$ are directed towards the downstream $\bar{x}$ and the normal $\bar{y}$ direction, respectively. The static pressure is denoted by $\bar{P}$, $\bar{\rho}$ is the density and $\bar{\mu}$ is the dynamic viscosity of the fluid.

For convenience, this set of second order differential equations is non-dimensionalized which involves the Reynolds number necessary for the following reduction of the equations.

The prescriptions for non-dimensionalization are:

$$u = \frac{\bar{u}}{V} = O(1)$$

$$v = \frac{\bar{v}}{V} = O(\varepsilon)$$

$$p = \frac{\bar{p}}{\bar{\rho} V^2} = O(1)$$

$$Re = \frac{\bar{\rho} L V}{\bar{\mu}} = O\left(\frac{1}{\varepsilon^2}\right)$$

$$x = \frac{\bar{x}}{L} = O(1)$$

$$y = \frac{\bar{y}}{L} = O(\varepsilon)$$

$V \rightarrow$ free stream velocity.

The pressure is non-dimensionalised by twice the dynamic pressure.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$(1)\frac{(1)}{(1)} \quad (\varepsilon)\frac{(1)}{(\varepsilon)} \quad (1) \quad \frac{1}{Re}\left(\frac{(1)}{(1)} \quad \frac{(1)}{(\varepsilon^2)}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

$$(1)\frac{(\varepsilon)}{(1)} \cdot (\varepsilon)\frac{(\varepsilon)}{(\varepsilon)} \quad (\varepsilon) \quad (\varepsilon)^2\left(\frac{(\varepsilon)}{(1)} \quad \frac{(\varepsilon)}{(\varepsilon^2)}\right).$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\frac{(1)}{(1)} \quad \frac{(\varepsilon)}{(\varepsilon)}$$

The boundary layer thickness δ is very small. So the distance y is very small compared to the length of the body L.

Consequently y is of the order ε which describes a value of V, therefore it is of the order 1. But the v-velocity component also has to be of the order ε. If the derivative $\partial u/\partial x$ is of the order 1 because x becomes, at its maximum, the length L, then the second term in the continuity equation $\partial v/\partial y$ has also to be of the order 1.

Now, with these assumptions the order of magnitude analysis can be done.

The viscous forces in the boundary layer can become of the same order of magnitude as the inertia forces only if the reynolds number is of the order of $1/\varepsilon^2$.

All terms of the normal momentum equation are of a smaller magnitude. This equation can only be in balance if the pressure term is of the same order of magnitude. Therefore, this equation delivers the information of negligible pressure gradient in the normal direction.

i.e. $$\frac{\partial p}{\partial y} = O(\varepsilon).$$

The derivation for the equation of motion at the outer edge of the boundary layer gives, if the inviscid velocity distribution is known

W.K.T,

$$dp = -\rho U dU$$

$$U dU = -\frac{1}{\rho} dP$$

Then,

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx}$$

By integrating above we obtain, Bernoulli's equation.

$$P + \frac{1}{2}\rho U^2 = C.$$

Summing up, by the order of magnitude analysis the Navier-Stokes equations and the continuity eqn have been simplified. They are known as 'Prandtl's boundary layer equations'.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

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The boundary conditions are;

on the surface.

$$y = 0; \quad u = 0; \quad v = 0.$$

on the outer edge of the boundary layer.

$$y = \delta = \frac{\bar{\delta}}{L}; \quad u = U(x).$$

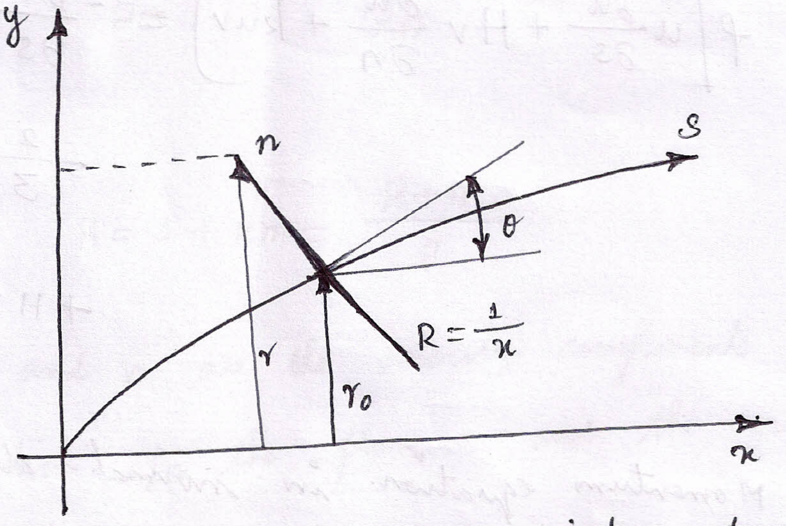
This set of equations is reduced by the unknown pressure P, when it becomes known from Bernoulli's equation, if only the inviscid velocity distribution $U(x)$ at the surface is provided.

Hierarchy of the Boundary layer equations

To develop a hierarchy of the fluid mechanical equations, the steady, compressible, laminar, two dimensional Navier-Stokes equations should be written for the Euclidian space in a layer close to the surface. This will say that a coordinate system, which may be surface oriented for a better adaption to the flow problem considered, is related to the cartesian coordinate system.

Both systems must be transferable from one to the other. If the Navier-Stokes equations can be formulated for such a surface oriented coordinate system, they will contain many additional terms due to the surface curvature. These terms can be understood as Coriolis and centripugal force terms caused by the change of the streamlines in downstream as well as in the cross flow direction depending on the curvature of the surface. Curvature induced terms will have different orders of magnitude. Some are important and others can be neglected depending on the specific flow problems.

A simple two-dimensional surface oriented coordinate system is fixed on an airfoil-like contour.



Surface oriented coordinate system

The relations between the new coordinate system and the cartesian one are.

$$x = \int_0^s \cos\theta(\kappa)\, ds - n \sin\theta(\kappa)$$

$$y = \int_0^s \sin\theta(\kappa)\, ds + n \cos\theta(\kappa).$$

The resultant set of differential equations due to the coordinate transformation consists of two equations of motion in the downstream direction s and the perpendicular direction n, the energy and the continuity equations.

Momentum equation in tangential direction.

$$\rho\left[u\frac{\partial u}{\partial s} + Hv\frac{\partial u}{\partial n} + kuv\right] = -\frac{\partial p}{\partial s} + \frac{\partial}{\partial s}\left[\frac{4\mu}{3H}\frac{\partial u}{\partial s} + \frac{4}{3}\frac{k\mu v}{H} - \frac{2}{3}\mu\frac{\partial v}{\partial n}\right] + H\frac{\partial}{\partial n}\left[\frac{\mu}{H}\frac{\partial v}{\partial s} + \mu\frac{\partial u}{\partial n} - \frac{\mu k u}{H}\right]$$

Momentum equation in normal direction.

$$\rho\left[u\frac{\partial v}{\partial s} + Hv\frac{\partial v}{\partial n} - ku^2\right] = -H\frac{\partial p}{\partial n} + H\frac{\partial}{\partial n}\left[\frac{4}{3}\mu\frac{\partial v}{\partial n} - \frac{2\mu}{3H}\frac{\partial u}{\partial s} - \frac{2}{3}\frac{\mu k v}{H}\right] + \frac{\partial}{\partial s}\frac{\mu}{H}\frac{\partial v}{\partial s} + \mu\frac{\partial u}{\partial n} - \frac{\mu k u}{H} + 2k\left[\mu\frac{\partial v}{\partial n} - \frac{\mu}{H}\frac{\partial u}{\partial s} - \frac{\mu k v}{H}\right]$$

Energy equation:

$$c_p\rho\left[u\frac{\partial T}{\partial s} + Hv\frac{\partial T}{\partial n}\right] = u\frac{\partial p}{\partial s} + Hv\frac{\partial p}{\partial n} + \frac{\partial}{\partial s}\left[\lambda\frac{\partial T}{\partial s}\right] + H\frac{\partial}{\partial n}\left[\lambda\frac{\partial T}{\partial n}\right] + H\frac{\mu}{2}\left\{\left[\frac{2}{H}\frac{\partial u}{\partial s} + \frac{2kv}{H}\right]^2 + \left[2\frac{\partial v}{\partial n}\right]^2 + 2\left[\frac{1}{H}\frac{\partial v}{\partial s} + \frac{\partial u}{\partial n} - \frac{ku}{H}\right]^2\right\} - \frac{2}{3}H\mu\left[\frac{1}{H}\frac{\partial u}{\partial s} + \frac{kv}{H} + \frac{\partial v}{\partial n}\right]^2$$

Continuity equation :

$$\frac{\partial(\rho u)}{\partial s} + \frac{\partial(H\rho v)}{\partial n} = 0$$

with

$$H = 1 + kn = \frac{R+n}{R}$$

Here u and v are the velocity components in the tangential direction of the flow s and the normal direction n, respectively.

$p \to$ Pressure
$\rho \to$ Density
$\mu \to$ Dynamic viscosity
$\lambda \to$ Thermal heat conductivity.

The curvature of the surface is involved in the geometrical coefficient H. This dimensional set of differential equations describe the laminar, compressible flow along arbitrary, two-dimensional curved surfaces. The order of magnitude of these quantities is,

$$s = \frac{\bar{s}}{L} = O(1) \;;\quad n = \frac{\bar{n}}{L} = O(\epsilon) \;;\quad k = \bar{k}L = O(1)$$

$$H = 1 + \bar{k}\bar{n} = O(1) \;;\quad u = \frac{\bar{u}}{u_\infty} = O(1) \;;\quad v = \frac{\bar{v}}{u_\infty} = O(\epsilon)$$

$$T = \frac{\bar{T}}{T_\infty} = O(1) \; ; \; p = \frac{\bar{p}}{\rho_\infty u_\infty^2} = O(1) \; ; \; \rho = \frac{\bar{\rho}}{\rho_\infty} = O(1)$$

$$\mu = \frac{\bar{\mu}}{\mu_\infty} = O(1) \; ; \; \lambda = \frac{\bar{\lambda}}{\lambda_\infty} = O(1) \; ; \; c_p = \frac{\bar{c}_p}{c_{p\infty}} = O(1)$$

Reynolds number $\quad Re = \dfrac{\rho_\infty u_\infty L}{\mu_\infty} = O\left(\dfrac{1}{\varepsilon^2}\right)$

Prandtl number $\quad Pr = \dfrac{c_{p\infty} \mu_\infty}{\lambda_\infty} = O(1)$

It is to be mentioned that the radius of curvature R is not allowed to be much larger than the characteristic length L, otherwise K would belong to another order of magnitude. The radius of curvature R is related to the curvature as follows

$$K = \bar{K} L = \frac{L}{R}.$$

When the radius R becomes very small compared to the length, H can exceed the order demanded above. To give an insight into the origin of the hierarchy of the boundary layer equations, the equations will be shown that retain terms only of the order $O(1)$ and $O(\varepsilon)$. The chosen equation is the tangential and normal momentum equation in dimensional form.

Order of magnitude $O(1)$

$$\rho\left[u\frac{\partial u}{\partial s} + Hv\frac{\partial u}{\partial n}\right] = -\frac{\partial p}{\partial s} + H\frac{\partial}{\partial n}\left(\mu\frac{\partial u}{\partial n}\right)$$

$$\frac{\partial p}{\partial n} = 0$$

$$\frac{\partial(\rho u)}{\partial s} + \frac{\partial(H\rho v)}{\partial n} = 0$$

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These equations are called the first order boundary layer equations. These equations become identical to Prandtl's boundary layer equations when the curvature goes to zero (H will be neglected).

Hence, Prandtl's equations are the lowest level of the hierarchy and they can be called as Zeroth order boundary layer equations.

Order of magnitude $O(\varepsilon)$

$$\rho\left[u\frac{\partial u}{\partial s} + Hv\frac{\partial u}{\partial n} + \frac{Kuv}{H}\right] = -\frac{\partial p}{\partial s} + H\frac{\partial}{\partial n}\left(\mu\frac{\partial u}{\partial n}\right) - K(u-\mu)\frac{\partial u}{\partial n}$$

$$\frac{\partial p}{\partial n} = \frac{K\rho u^2}{H}$$

The centrifugal term in left hand side and the dissipative terms due to curvature in right hand side are added. The pressure gradient normal to the flow is no longer zero. These are called second order boundary layer equations.

Transformation of the boundary layer equations

The boundary equations for the flow around a body of ~~the~~ revolution without inclination are considered. The flow is two dimensional because it does not vary in the circumferential direction. The first order laminar compressible boundary layer equations in dimensional form are,

continuity equation:

$$\frac{\partial}{\partial s}(r^j \rho u) + \frac{\partial}{\partial n}(r^j H \rho v) = 0$$

Downstream momentum equation:

$$\frac{\rho u}{H}\frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial n} = -\frac{1}{H}\frac{\partial p}{\partial s} + \frac{\partial}{\partial n}\left(\mu \frac{\partial u}{\partial n}\right)$$

Energy equation:

$$\frac{C_p \rho u}{H}\frac{\partial T}{\partial s} + C_p \rho v \frac{\partial T}{\partial n} = \frac{u}{H}\frac{\partial p}{\partial s} + \mu\left(\frac{\partial u}{\partial n}\right)^2 + \frac{\partial}{\partial n}\left(\lambda \frac{\partial T}{\partial n}\right)$$

With $H = 1 + kn$.

For $j = 0$ the flow is purely two dimensional and for $j = 1$ it is axisymmetric.

These equations present a system of coupled partial differential equations depending on the spatial directions s and n. Their character is parabolic and therefore one has an initial - boundary value problem that can be solved by a marching procedure.

Often transformations help to simplify the governing equations concerning the solution technique. By a transformation the boundary layer in the transformed plan can be kept at a nearly uniform thickness for many flow situations. Such a transformation, which is called a 'compressibility and similarity transformation', was first proposed by Levy-Lees. It reads for axisymmetric bodies:

$$\xi(s) = \int_0^s \rho_e \mu_e u_e R^{2j} \, ds$$

$$\eta(s, n) = \frac{\rho_e \mu_e}{\sqrt{2\xi}} \int_0^N \frac{\rho}{\rho_e} r^j \, dn$$

The index e denotes the values at the outer edge of the boundary layer flow and R denotes the local radius of a body of revolution. Introducing the transformation rules gives

continuity equation:

$$2\xi \frac{\partial F}{\partial \xi} + \frac{\partial V}{\partial \eta} + F = 0$$

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$$\frac{2\xi}{H}F\frac{\partial F}{\partial \xi}+\frac{V}{H}\frac{\partial F}{\partial \eta} = -\frac{2\xi F^2}{H u_e}\frac{du_e}{d\xi}+\frac{\partial}{\partial \eta}\left[\left(\frac{r}{R}\right)^{2j}\frac{\rho\mu}{\rho_\infty \mu_\infty}\frac{\partial F}{\partial \eta}\right]$$

Energy equation:

$$\frac{2\xi}{H}F\frac{\partial S}{\partial \xi}+\frac{V}{H}\frac{\partial S}{\partial \eta} = \frac{2\xi F}{H T_e}\frac{dT_e}{d\xi}S+\left(\frac{r}{R}\right)^{2j}\frac{\rho\mu}{\rho_\infty \mu_\infty}\frac{u_e^2}{C_p T_e}\left(\frac{\partial F}{\partial \eta}\right)^2$$

$$+\frac{\partial}{\partial \eta}\left[\left(\frac{r}{R}\right)^{2j}\frac{\rho\mu}{\rho_\infty \mu_\infty}\frac{1}{Pr}\frac{\partial S}{\partial \eta}\right]$$

With $F=\frac{u}{u_e}$, $S=\frac{T}{T_e}$,

V represents the transformed velocity component.

$$V=\frac{2\xi}{\rho_e u_e \mu_e R^{2j}}\left[F\left(\frac{\partial \eta}{\partial s}+\rho v\frac{r^j}{\sqrt{2\xi}}\right)\right]$$

For the case of vanishing ξ near the sharp tip of the body a singularity arises, but this makes the initial conditions for the transformed equations easy to calculate because the ξ-derivatives drop out of the set of equations. Once an initial rough guess of F and S is done, the transformed normal velocity V can easily be integrated from the continuity equation. Iterating steps will correct the initial guesses of F and S.

The disadvantage is that, apart from their complicated form, the total number of grid points in the normal direction n has to be calculated starting right away from the initial profile although here the boundary layer is very thin.

On the contrary, the calculation in the physical plane needs very few grid points at the beginning. The numbers must be continuously increased due to the growth of the boundary layers. The disadvantage of this method is to correctly overcome the singularity at the sharp tip where S equals zero.

Numerical Solution Method.

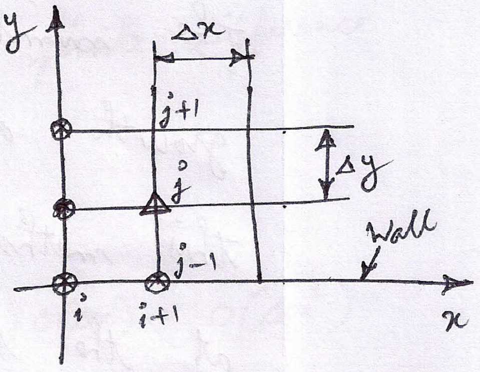
Choise of discretization model.

To come to a numerical solution of a set of partial differential equations it is usual to replace the differential quotients by finite difference quotients taking into account that a truncation error of a certain order of magnitude.

By rearranging the finite difference equations a system of algebraic equations is obtained which can be solved by means of the known methods.

Parabolic equations as observed have a first order differential in the marching direction. As the flow is not allowed to reverse, the values of each quantity at the last upstream grid line normal to the surface are known.

Consider a grid as shown in figure, where Δx and Δy are the step sizes in the tangential and normal direction to the surface, the known points are on the left-hand side and the unknown on the right.

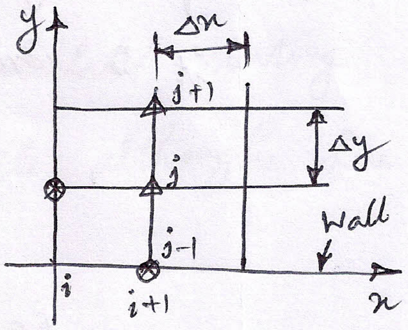


Grid for an explicit method.

Also the boundary conditions at the wall are given. Therefore, it is easy to calculate the flow quantities at the point. Because of the direct calculation of only one point on the grid line, this is called an <u>explicit method</u>. The explicit method causes strong restrictions in the choice of the downstream step size.

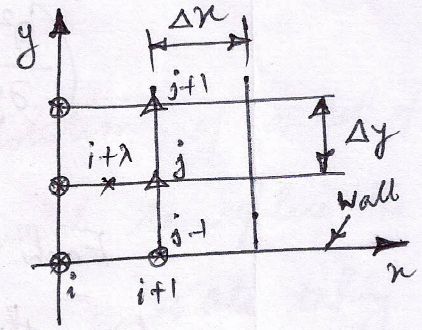
Fully implicit method is an another extreme choice of a computational grid.

Only one known grid point from the preceding step is used, while on the actual one all points are unknown except the boundary values. This method is, concerning the choice of the step size, unconditionally stable but may lead to a poor accuracy. If there is no restriction on the step size in the downstream direction it becomes a fast calculation method.



Grid for a fully implicit method

To formulate something in between these extremes which will result in both a fast and accurate solution. Crank-Nicholson proposed a computational mesh method. Here, all points of the known and unknown grid lines are involved, but the centre of discretization is located at the point $i+\lambda$.



Grid for a Generalised implicit method.

$\lambda = 1/2$ was originally proposed by Crank-Nicholson.

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Generalised Crank - Nicholson Scheme.

In order to analyse the stability and accuracy of a generalization of the Crank - Nicholson scheme, it is convenient to utilize the linear model eqn.

$$\frac{\partial \phi}{\partial x} = a \frac{\partial^2 \phi}{\partial y^2} \longrightarrow (1)$$

Equation (1) is discretized around the mesh point $(i+\lambda, j)$, with λ ranging between 0 and 1. For $\lambda = 0$ an explicit scheme is recovered, while $\lambda = 1$ corresponds to the fully implicit case. If the grid is uniform, the x- derivative is approximated by the finite difference relation.

$$\left(\frac{\partial \phi}{\partial x}\right)_{i+\lambda, j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + \left(\lambda - \frac{1}{2}\right) O(\Delta x) + O(\Delta x^2)$$

and the y - derivative is replaced by the weighted mean

$$\left(\frac{\partial^2 \phi}{\partial y^2}\right)_{i+\lambda, j} = \lambda \left(\frac{\partial^2 \phi}{\partial y^2}\right)_{i+1, j} + (1-\lambda)\left(\frac{\partial^2 \phi}{\partial y^2}\right)_{i,j}$$

Each second - order derivative is then replaced by the usual three - point centred finite difference relation:

Then equation (1) ~~can~~ can be written as,

$$\frac{\phi_{i+1,j}-\phi_{i,j}}{\Delta x}=a\left[\lambda\left(\frac{\phi_{i+1,j+1}-2\phi_{i+1,j}+\phi_{i+1,j-1}}{\Delta y^2}\right)+(1-\lambda)\left(\frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{\Delta y^2}\right)\right]$$

$$\phi_{i+1,j}-\phi_{i,j}=\frac{a\Delta x}{\Delta y^2}\left[\lambda\left(\phi_{i+1,j+1}-2\phi_{i+1,j}+\phi_{i+1,j-1}\right)+(1-\lambda)\left(\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}\right)\right]$$

Which can be recast as.

$$\lambda\frac{a\Delta x}{\Delta y^2}\,\phi_{i+1,j+1}-\left(1+2\lambda\frac{a\Delta x}{\Delta y^2}\right)\phi_{i+1,j}+\lambda\frac{a\Delta x}{\Delta y^2}\,\phi_{i+1,j-1}=D_j$$

Where,

D_j a function of ϕ computed at station i.

Discretization of the Boundary layer equations.

The first order laminar compressible boundary layer equations are non-dimensionalised.

continuity eqn.

$$\frac{\partial}{\partial s}(r^j\rho u)+\frac{\partial}{\partial n}(r^j H\rho v)=0\longrightarrow (1)$$

Downstream momentum equation

$$\frac{\rho u}{H}\frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial n} = -\frac{1}{H}\frac{\partial p}{\partial s} + \frac{1}{Re}\frac{\partial}{\partial n}\left(\mu \frac{\partial u}{\partial n}\right) \quad (2)$$

Energy equation

$$\frac{C_p \rho u}{H}\frac{\partial T}{\partial s} + C_p \rho v \frac{\partial T}{\partial n} = Ec\,\frac{u}{H}\frac{\partial p}{\partial s} + \frac{Ec}{Pr}\,\mu\left(\frac{\partial u}{\partial n}\right)^2 + \frac{1}{RePr}\frac{\partial}{\partial n}\left(\lambda \frac{\partial T}{\partial n}\right) \quad (3)$$

where, $H = 1 + kn$.

Since, the velocity u_e is given from measurements or inviscid flow calculations eqn (2) can be developed at the point $n = \delta$.

$$\rho_e u_e \frac{\partial u_e}{\partial s} = -\frac{\partial p}{\partial s}$$

The demand of constant total enthalpy in the outer flow yields the boundary conditions for the temperature T_e:

$$T_e = 1 + Ec\,(1 - u_e^2)$$

The conditions for the velocity at the wall are the no-slip assumption

$$u_w = 0$$

Ec = Eckert Number

$= \frac{\text{Inertia force}}{\text{change in Enthalpy}}$

and the zero normal velocity statement

$$V_w = 0$$

For the wall temperature T_w, according to the demands a distribution can be prescribed or the wall may be adiabatic.

$$T_w = T_w(s) \qquad \text{(prescribed)}$$

$$\left(\frac{\partial T}{\partial n}\right)_w = 0 \qquad \text{(adiabatic)}$$

Since eqn ① to ③ contains four unknowns, to close the system, the ideal gas equation is added.

$$p = \rho R T$$

Now the discretization of the downstream momentum equation ~~was~~ evaluated from the generalised Crank-Nicholson scheme.

First-order derivative in downstream direction:

$$\left(\frac{\partial u}{\partial s}\right)_{i+\lambda, j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta s} \longrightarrow (4)$$

First-order derivative in normal direction:

$$\left(\frac{\partial u}{\partial n}\right)_{i+\lambda, j} = \lambda\left(\frac{\partial u}{\partial n}\right)_{i+1,j} + (1-\lambda)\left(\frac{\partial u}{\partial n}\right)_{i,j} \longrightarrow (5)$$

The normal derivative at $i+1, j$ is a second-order centred difference solution

$$\left(\frac{\partial u}{\partial n}\right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta n}$$

The second term on the right-hand side of eqn ②.

$$\frac{\partial}{\partial n}\left(\mu \frac{\partial u}{\partial n}\right) = \mu \frac{\partial^2 u}{\partial n^2} + \frac{\partial \mu}{\partial n}\frac{\partial u}{\partial n}$$

the dynamic viscosity is an analytical function of T.

$$\frac{\partial}{\partial n}\left(\mu \frac{\partial u}{\partial n}\right) = \mu \frac{\partial^2 u}{\partial n^2} + \frac{\partial \mu}{\partial T}\frac{\partial T}{\partial n}\frac{\partial u}{\partial n}$$

now the second derivative becomes,

$$\left(\frac{\partial^2 u}{\partial n^2}\right)_{i+\lambda, j} = \frac{1}{\Delta n^2}\Big[\lambda\left(u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}\right)$$

$$+ (1-\lambda)\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right)\Big] \quad ⑥$$

The non-linear coefficient of $\frac{\partial u}{\partial s}$ is discretized as,

$$\left(\frac{\rho u}{H}\right)_{i+\lambda, j} = \frac{\rho_{i,j}\, u_{i,j}}{\left(\lambda H_{i+1,j} + (1-\lambda) H_{i,j}\right)} = A.$$

The other coefficients are,

$$(\rho v)_{i+\lambda, j} = \rho_{i,j}\, v_{i,j} = B$$

$$\mu_{i+\lambda, j} = \mu_{i,j} = C$$

$$\left(\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial n}\right)_{i+\lambda,j} = \left(\frac{\partial u}{\partial T}\right)_{i,j}\left(\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta n}\right) = D$$

The pressure is only a function of the downstream coordinate s and it is known from the inviscid flow calculation.

$$-\frac{1}{H}\frac{\partial P}{\partial s} = -\left[\lambda\left(\frac{1}{H}\frac{\partial P}{\partial s}\right)_{i+1,j} + (1-\lambda)\left(\frac{1}{H}\frac{\partial P}{\partial s}\right)_{i,j}\right] = E$$

Now the equation (2) can be rewritten as,

$$A\frac{\partial u}{\partial s} + B\frac{\partial u}{\partial n} = -E + C\frac{\partial^2 u}{\partial n^2} + D\frac{\partial u}{\partial n}$$

Then,

$$A\frac{\partial u}{\partial s} + B\frac{\partial u}{\partial n} - C\frac{\partial^2 u}{\partial n^2} - D\frac{\partial u}{\partial n} + E = 0 \quad \rightarrow (7)$$

Substituting eqn (4) to (6) in (7) yields.

$$\frac{\lambda}{2\Delta n}\left[B - \frac{2C}{Re\Delta n} - D\right]u_{i+1,j+1} + \left[\frac{A}{\Delta s} + \frac{2\lambda C}{Re\,\Delta n^2}\right]u_{i+1,j}$$

$$+ \frac{\lambda}{2\Delta n}\left[-B - \frac{2C}{Re\Delta n} + D\right]u_{i+1,j-1} = F \quad \rightarrow (8)$$

In abbreviated form,

$$a_j\, u_{i+1,j+1} + b_j\, u_{i+1,j} + c_j\, u_{i+1,j-1} = d_j \quad \rightarrow (9)$$

The energy equation takes the corresponding form.

$$e_j T_{i+1,j+1} + f_j T_{i+1,j} + g_j T_{i+1,j-1} = h_j \rightarrow (10)$$

Then the continuity eqn is,

$$p_j V_{i+1,j+1} + q_j V_{i+1,j} + r_j V_{i+1,j-1} = S_j \rightarrow (11)$$

The only unknown which is not yet treated is the density, ρ. The perfect gas law fills this gap. First order boundary layer theory does not know a pressure variation.

$$\rho T = C$$

It yields the discretized form as.

$$\rho_{i+1,j+1} = \rho_{i+1,j} \frac{T_{i+1,j}}{T_{i+1,j+1}} \rightarrow (12)$$

The procedure may be described as follows

i) Solution of the momentum and the energy equation until convergence is achieved.

ii) Solution of the continuity equation.

iii) Calculation of the density.

iv) Repeat of the previous steps until convergence of the equations is achieved.

v) Start the calculations at the next downstream position.

Stability properties of Explicit and Implicit methods.

The outcome of the competetion between explicit and implicit methods is governed by their respective stability properties. The space-discretization of a time-dependent partial differential equation produces a system of ordinary differential equations. Let us consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(1)}.$$

After discretization in space using central finite differences, we obtain the following system of ordinary differential equations.

$$\frac{du_i}{dt} = \frac{1}{\Delta x^2}\left(u_{i-1} - 2u_i + u_{i+1}\right)$$

$$\frac{dU}{dt} = AU + F(t)$$

with,

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} ; \quad A = \frac{1}{\Delta x^2}\begin{bmatrix} \text{B.C.} & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & \text{B.C.} \end{bmatrix}$$

Therefore, the ~~analysis~~ analysis of the stability of a time stepping scheme for solving the PDE reduces to the analysis of the stability of a time stepping scheme for solving a system of ODEs.

Furthermore, when we consider a periodic solution in space, i.e. $u = u(t)\, e^{i k_m x}$, the system of ODEs reduces to a single ODE. Indeed, inserting the periodic solution hypothesis give,

$$u_{i+1} = u(t)\, e^{i k_m x_{i+1}} = u(t)\, e^{i k_m (x_i + \Delta x)} = u(t)\, e^{i k_m x_i}\, e^{i k_m \Delta x}$$

$\therefore$

$$u_{i+1} = u_i\, e^{i k_m \Delta x}.$$

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$$u_{i-1} = u_i\, e^{-i k_m \Delta x}$$

Then, we obtain

$$\frac{du_i}{dt} = \left[\frac{e^{i k_m \Delta x} - 2 + e^{-i k_m \Delta x}}{\Delta x^2}\right] u_i = -\frac{4}{\Delta x^2} \sin^2\left(\frac{k_m \Delta x}{2}\right) u_i$$

i.e. an ODE whose coefficient q depends on the reduced wavenumber $k_m \Delta x$, the locus of q being called the Fourier footprint of the discretized equation.